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On the Polynomials Orthogonal on Regular Polygons

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The two-parameter Pastro–Al-Salam–Ismail (PASI) polynomials are known to be bi-orthogonal on the unit circle with continuous weight function when $0<q<1$. We study the case of q a root of unity. It is shown that corresponding PASI polynomials are orthogonal on the unit circle with discrete measure located on the vertices of the regular N-gon. Cases leading to a positive weight function are analyzed. In particular, we obtain trigonometric analogs of the Askey–Szegő polynomials which are orthogonal on regular N-gons with positive weight function. © 1999 Academic Press

1. BI-ORTHOGONAL POLYNOMIALS AND THEIR SIMPLEST PROPERTIES

The polynomials which are orthogonal on the unit circle were introduced by Szegő [21] and are known to possess many interesting properties analogous to those for the polynomials orthogonal on the real interval.

One of their main properties is the recurrence relation

$$P_{n+1}(z)=zP_n(z)-\alpha_n P_n^*(z), \qquad P_0(z)=1, \tag{1.1}$$

where $\alpha_n=-P_{n+1}(0)$ are so-called reflection (recurrence) parameters and the polynomial $P_n^*(z)$ is defined as $P_n^*(z)=z^n\bar{P}_n(1/z)$. It can be shown (see, e.g., [10, 13]) that if the recurrence coefficients satisfy the condition $|\alpha_n|<1$ then there exists a unique finite positive Borel measure μ on the unit circle such that the polynomials $P_n(z)$ are orthogonal:

$$\int_0^{2\pi} P_n(e^{i\theta})\,\bar{P}_m(e^{-i\theta})\,d\mu(\theta)=h_n\,\delta_{nm}. \tag{1.2}$$

There is an interesting extension of the recurrence relation (1.1) proposed by G. Baxter [5]. This extension involves two sets of the polynomials $P_n(z)$ and $Q_n(z)$,

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$$P_{n+1}(z) = zP_n(z) - \alpha_n z^n Q_n(1/z)$$
$$Q_{n+1}(z) = zQ_n(z) - \beta_n z^n P_n(1/z) \tag{1.3}$$
$$P_0(z) = Q_0(z) = 1,$$

where α_n, β_n are (complex) recurrence coefficients. It is expected that under some conditions for recurrence coefficients the recurrence system (1.3) leads to the polynomials which are *biorthogonal* with some measure on the unit circle.

The pair of polynomials $P_n(z)$, $Q_n(z)$ is closely connected with so-called Laurent orthogonal polynomials [15, 17, 18]. Namely, one can introduce moments c_n defined for all integers n through some linear Laurent functional $\mathscr{L}\{z^n\} = c_n$, $n = 0, \pm 1, \pm 2, \ldots$. Then one can construct polynomials $P_n(z)$ satisfying the orthogonality property

$$\mathscr{L}\{z^{-j} P_n(z)\} = 0, \qquad 0 \leqslant j < n. \tag{1.4}$$

It is easily shown that the orthogonality relation (1.4) can be rewritten in terms of the bi-orthogonality relation

$$\mathscr{L}\{P_n(z)\, Q_m(1/z)\} = 0, \qquad m \neq n, \tag{1.5}$$

where the polynomials $P_n(z)$ and $Q_m(z)$ satisfy the recurrence relations (1.3). Vice versa, it can be proven [15] that the recurrence relations (1.3) lead to existence of a linear Laurent functional $\mathscr{L}$ providing the bi-orthogonality condition (1.5) (an analogue of the Favard theorem for the Laurent polynomials).

More general systems of bi-orthogonal *rational functions* (instead of polynomials) are currently being intensively studied (see, e.g., [6, 16], where many interesting results including analogues of the Favard theorem are contained).

In [1, 18] a concrete two-parameter system of polynomials (satisfying (1.3)) which are bi-orthogonal on the unit circle was studied. The orthogonality relation for this system arises from a special kind of Ramanujan q-beta integral. The corresponding polynomials are expressed in terms of basic hypergeometric functions (see the next section) with the base q satisfying the condition $0 < q < 1$. When two parameters of the polynomials coincide with one another, then so-called Askey–Szegő polynomials on the unit circle [2] are obtained.

An interesting problem arises if we ask what happens when q becomes a root of unity: $q^N = 1$. Then it can be easily shown that only finite-dimensional polynomials are possible, because otherwise the basic hypergeometric function is not defined for $n > N$. Hence the corresponding measure should contain only a finite number of points of increase. In this case we need general theorems concerning finite bi-orthogonality for the polynomials defined by the recurrence system (1.3). Earlier we considered finite orthogonality for the Askey–Wilson polynomials for q a root of unity [19].

Our preliminary result will be an analog of the Christoffel–Darboux formula.

PROPOSITION 1. *Let $P_n(z)$, $Q_n(z)$ be two systems of the polynomials satisfying the recurrence system* (1.3) *with arbitrary complex coefficients α_n, β_n. Then the following identity takes place,*

$$\frac{P_{n+1}(x)\, Q_n(1/y) - (x/y)^n\, P_{n+1}(y)\, Q_n(1/x)}{h_n} = (x-y) \sum_{k=0}^{n} \frac{P_k(x)\, Q_k(1/y)}{h_k}, \tag{1.6}$$

where

$$h_n = \prod_{k=0}^{n-1} (1 - \alpha_k \beta_k), \qquad h_0 = 1. \tag{1.7}$$

Proof. Denote

$$A_{-1} = 0, \qquad A_k(x, y) = y^{-k} \frac{P_{k+1}(x)\, Q_k^*(y) - P_{k+1}(y)\, Q_k^*(x)}{h_k},$$

$$k = 0, 1, 2, ...,$$

where (by definition) $Q_k^*(x) = x^k Q_k(1/x)$. From the recurrence relations (1.3) we have

$$A_k(x, y) - A_{k-1}(x, y) = (x-y) \frac{P_k(x)\, Q_k(1/y)}{h_k}. \tag{1.8}$$

Then summing (1.8) from $k=0$ to $k=n$ we arrive at the identity (1.6).

In fact the formula (1.6) was derived by G. Baxter [5] in a somewhat different form (see his formula (2.5)).

The formula (1.6) is sufficient to state the following general theorem concerning the finite bi-orthogonality property:

THEOREM 1. *Under the conditions of Proposition* 1 *assume additionally that*

(i) $h_n \neq 0$, $n = 0, 1, ..., N-1$; $h_N = 0$ (*i.e.*, $\alpha_{N-1} \beta_{N-1} = 1$);

(ii) *all the zeros z_s, $s = 0, 1, ..., N-1$ of the polynomial $P_N(z)$ are simple.*

Then the following bi-orthogonality relations take place,

$$\sum_{s=0}^{N-1} w_s P_n(z_s)\, Q_m(1/z_s) = h_n\, \delta_{mn}, \tag{1.9}$$

$$\sum_{s=0}^{N-1} \tilde{w}_s P_n(1/z_s)\, Q_m(z_s) = h_n\, \delta_{mn}, \qquad n, m = 0, 1, ..., N-1, \tag{1.10}$$

where the weight functions have the expressions

$$w_s = \frac{h_{N-1}}{Q_{N-1}(1/z_s)\, P'_N(z_s)}, \tag{1.11}$$

$$\tilde{w}_s = \frac{h_{N-1}}{P_{N-1}(z_s)\, Q'_N(1/z_s)}. \tag{1.12}$$

Proof. From the Christoffel–Darboux identity (1.6) and its convolution form (i.e., when $x \to y$) we get the relation

$$\sum_{k=0}^{N-1} \frac{P_k(z_s)\, Q_k(1/z_t)}{h_k} = \rho_s\, \delta_{st}, \qquad k, s, t = 0, 1, ..., N-1, \tag{1.13}$$

where

$$\rho_s = \frac{Q_{N-1}(1/z_s)\, P'_N(z_s)}{h_{N-1}} \tag{1.14}$$

and z_s are (simple) zeros of the polynomial $P_N(z)$. By condition (i) of the theorem $h_k \neq 0$, $k = 0, 1, ..., N-1$. Moreover $\rho_s \neq 0$, $s = 0, 1, ..., N-1$. Indeed, $P'_N(z_s) \neq 0$ because all the zeros z_s are simple; if one assumes that $Q_{N-1}(1/z_s) = 0$ then it is easy to show from the recurrence relation (13) that $P_{N-1}(z_s) = P_{N-2}(z_s) = \cdots = P_0(z_s) = 1 = 0$, which is impossible. Hence $\rho_s \neq 0$, $s = 0, 1, ..., N-1$. This allows us to introduce two $N \times N$ matrices, A and B, with entries $A_{sk} = P_k(z_s)/h_k$, $B_{ks} = Q_k(1/z_s)/\rho_s$; $k, s = 0, 1, ..., N-1$. The relation (1.13) can be rewritten in matrix form as

$$AB = I, \tag{1.15}$$

where I is the identity matrix. The relation (1.15) means that both matrices A and B are nondegenerate and that they are reciprocal to one another. Hence $BA = I$, which is equivalent to the orthogonality relation (1.9). The relation (1.10) is obtained from the symmetry between polynomials $P_n(z)$ and $Q_n(z)$ and from the observation that under the condition $h_N = 0$ the zeros of the polynomial $Q_N(z)$ coincide with $1/z_s$, $s = 0, 1, ..., N-1$. Hence the theorem is proven.

Remark. The relation (1.9) provides a concrete realization of the Laurent functional $\mathcal{L}$ and bi-orthogonality relation (1.6). The additional condition (ii) is essential, because for generic (complex) recurrence coefficients α_n, β_n the polynomials $P_N(z)$ may have multiple zeros. Hence in any concrete case one should first verify whether the condition (ii) is satisfied. In the next section we check this condition using an explicit expression for corresponding polynomials $P_N(z)$. The question under which restrictions upon the coefficients α_n, β_n the zeros are simple is an interesting open problem. One such restriction is well known: $\beta = \bar{\alpha}_n$, $|\alpha_n| < 1$. This restriction corresponds just to the polynomials (1.1) orthogonal on the unit circle (for the proof that all zeros are simple see, e.g., [4, 13]).

COROLLARY 1. *If* $\alpha_n = \beta_n$, $n = 0, 1, ..., N-1$, *then* $Q_n(z) = P_n(z)$, *the zeros of the polynomial* $P_N(z_s)$ *are symmetric* (*i.e., for any root* z_s *there exists the root* $1/z_s$), *and two bi-orthogonal relations* (1.9), (1.10) *are reduced to the orthogonal relation*

$$\sum_{s=0}^{N-1} w_s P_n(z_s)\, P_m(1/z_s) = h_n\, \delta_{nm}, \tag{1.16}$$

where the weight function is

$$w_s = \frac{h_{N-1}}{P_{N-1}(1/z_s)\, P'_N(z_s)}. \tag{1.17}$$

COROLLARY 2. *If* $\beta_n = \bar{\alpha}_n$ *then* $Q_n(z) = \bar{P}_n(z)$ *and the recurrence system* (1.3) *coincides with* (1.1). *The orthogonality relation in this case is written as*

$$\sum_{s=0}^{N-1} w_s P_n(z_s)\, \bar{P}_m(1/z_s) = h_n\, \delta_{nm}, \tag{1.18}$$

where

$$w_s = \frac{h_{N-1}}{\bar{P}_{N-1}(1/z_s)\, P'_N(z_s)}. \tag{1.19}$$

Remark. In Corollary 2 the recurrence coefficients α_n need not satisfy the condition $|\alpha_n| < 1$. Hence the zeros of the polynomial $P_N(z)$ need not lie on the unit circle. For example, the zeros may be located on the real interval [18, 20, 22]. The orthogonality relation (1.19) is valid for all possible locations of the zeros (with the only assumption that all the zeros are simple).

In the next sections we apply these general results to derive concrete examples of the orthogonality relation for q a root of unity. In all examples presented the zeros lie at the vertices of a regular N-gon. Hence we get non-trivial examples of polynomials which are (bi) orthogonal on regular polygons. Perhaps these examples are new. Note that polynomials in *two variables* orthogonal on the interior of regular polygons with continuous measure were considered by Dunkl [9].

2. THE ZEROS AND ORTHOGONALITY FOR PASTRO–AL-SALAM–ISMAIL POLYNOMIALS

In [18] a very interesting system of polynomials $P_n(z)$, $Q_n(z)$ (containing two real parameters) which are bi-orthogonal on the unit circle was discovered. In [1] this system was slightly generalized to generic complex parameters. We will call corresponding objects Pastro–Al-Salam–Ismail (PASI) polynomials.

The recurrence coefficients of the system (1.3) are defined as

$$\alpha_n = -q^{(n+1)/2} \frac{(b;q)_{n+1}}{(aq;q)_{n+1}}, \qquad \beta_n = -q^{(n+1)/2} \frac{(a;q)_{n+1}}{(bq;q)_{n+1}}, \tag{2.1}$$

where a, b are arbitrary complex parameters and q is a real parameter such that $|q|<1$, and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the q-shifted factorial.

The explicit expression for the polynomials $P_n(z)$, $Q_n(z)$ is [1]

$$P_n(z;a,b) = q^{n/2} \frac{(b;q)_n}{(aq;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k\,(aq;q)_k}{(q;q)_k\,(q^{1-n}b^{-1};q)_k} (zq^{1/2}/b)^k, \tag{2.2}$$

$$Q_n(z;a,b) = P_n(z;b,a). \tag{2.3}$$

Al-Salam and Ismail showed [1] that under the conditions $|aq^{1/2}|<1$ and $|bq^{1/2}|<1$ the polynomials $P_n(z)$ and $Q_n(z)$ are bi-orthogonal on the unit circle with some continuous measure.

In this section we consider the case where q is a primitive Nth root of unity, i.e.,

$$q = \exp(2i\pi M/N), \tag{2.4}$$

where M and N are two co-prime integers ($M<N$). We will assume also that M is *odd* (in particular, we will consider also the simplest case $M=1$).

The condition (i) of Theorem 1 is fulfilled provided a^N, b^N, $(ab)^N \neq 1$. Assuming that these inequalities are valid let us find the zeros of the polynomial $P_N(z)$. For this goal note that the explicit expression (2.2) is valid for $n=0, 1, ..., N-1$. For $n=N$ this expression is not correct because $(q^{-N}; q)_N=(q; q)_N=0$ and hence we have indeterminacy in the last term of the sum. However, this indeterminacy can be avoided by the limiting procedure: put $q=\exp(\varepsilon+2i\pi M/N)$ and take the limit $\varepsilon\to 0$. Then

$$\lim_{\varepsilon\to 0}\frac{(q^{-N}; q)_N}{(q; q)_N}=-1$$

whereas other terms in the sum (2.2) (except of first one) are zero. So we have for $P_N(z)$ a simple expression

$$P_N(z)=z^N-\frac{1-b^N}{1-a^N} \tag{2.5}$$

(in (2.5) we used an obvious formula $(a; q)_N=1-a^N$).

In what follows we restrict ourselves to the relation

$$b=aq^j, \qquad j=0, 1, ..., N-1. \tag{2.6}$$

Then the zeros of the polynomials $P_N(z)$ and $Q_N(z)$ coincide with the roots of unity

$$z_s=q^s, \qquad s=0, 1, ..., N-1 \tag{2.7}$$

and obviously

$$P'_N(z_s)=Q'_N(z_s)=Nq^{-s}. \tag{2.8}$$

The most non-trivial part of the weight function (1.11) is evaluation of $Q_{N-1}(1/z_s)$ which is reduced to calculation of the following sum:

$$S(s, j)=\sum_{k=0}^{N-1}\frac{(aq^{1+j}; q)_k}{(q^2/a; q)_k}(q^{1/2-s}/a)^k. \tag{2.9}$$

Lemma 1. *The sum $S(s, j)$ is*

$$S(s, j)=A_j\frac{(q^{3/2}/a; q)_s\, a^s}{(aq^{1/2}; q)_s\,(aq^{-s-1/2}; q)_{j+1}}, \tag{2.10}$$

where

$$A_j=-q^{1/2}\frac{(-q^{1/2}; q^{1/2})_{N-1}\,(a; q^{1/2})_{N-1}\,(a^2/q; q)_{j+1}}{(a; q)_{j+1}\,(a; q)_{N-1}}. \tag{2.11}$$

The proof of this lemma is contained in the Appendix.

From this lemma we immediately find the expression for the weight function

$$w_s = B\,\frac{(aq^{1/2};q)_s\,(aq^{-1/2-s};q)_{j+1}}{(q^{3/2}/a;q)_s}\,(q/a)^s,$$

$$s = 0, 1, ..., N-1, \quad j = -1, 2, ..., N-2, \tag{2.12}$$

where

$$B = \frac{(1+a^N)(1-a)(a;q)_{j+1}}{(a^2/q;q)_{j+2}\,(-q^{1/2};q^{1/2})_{N-1}\,(a;q^{1/2})_{N-1}}. \tag{2.13}$$

The formula (2.12) is valid for all complex values of a except roots of unity: $a^N \neq 1$. In particular, for $a \to 0$ we get the case of Rogers–Szegő polynomials with the recurrence coefficients

$$\alpha_n = \beta_n = -q^{(n+1)/2}.$$

In this limit the weight function (2.12) becomes

$$w_s = (-1)^s\,\frac{q^{-s^2/2}}{(-q^{1/2};q^{1/2})_{N-1}}.$$

The normalization condition $\sum_{s=0}^{N-1} w_s = 1$ leads to the identity

$$\sum_{s=0}^{N-1} (-1)^s\,q^{-s^2/2} = (-q^{1/2};q^{1/2})_{N-1}. \tag{2.14}$$

The identity (2.14) was firstly derived by Gauss in 1811; it represents the transformation for the famous “Gauss sum” [7]. The same Gauss sum arises also in the theory of the Askey–Wilson polynomials for q a root of unity (see, e.g., [19]).

3. THE CASE OF REAL POLYNOMIALS WITH POSITIVE WEIGHT FUNCTION

So far, we considered the case when a is an arbitrary complex parameter. In this section we consider a special case where $a = q^\gamma$, where γ is a real parameter satisfying the inequality

$$-1/2 < \gamma < 1/2. \tag{3.1}$$

We restrict ourselves with the symmetric case $j=0$. Then both the recurrence coefficients are real and coincide with one another

$$\alpha_n=\beta_n=-\frac{\sin(\pi M\gamma/N)}{\sin(\pi M(n+\gamma+1)/N)}. \tag{3.2}$$

Moreover, if one chooses $M=1$ then $|\alpha_n|<1$, $n=0, 1, ..., N-2$. Hence, by the "Favard theorem" [10, 13] we conclude that the weight function w_s for the corresponding polynomials $P_n(z)$ should be positive, $w_s>0$, $s=0, 1, ..., N-1$. Moreover, it is obvious from the reality of α_n that the polynomials $P_n(z)$ have real coefficients.

Thus we obtain that the polynomials $P_n(z)$ are orthogonal on the regular N-gon

$$\sum_{s=0}^{N-1} w_s P_n(q^s)\, P_m(q^{-s})=h_n\,\delta_{mn}. \tag{3.3}$$

After some manipulations, the expression (2.12) for the weight function w_s can be transformed in our case to a more attractive form

$$w_s=B_N \sin\omega(s-\gamma+1/2) \prod_{k=1}^{s} \frac{\sin\omega(k+\gamma-1/2)}{\sin\omega(k-\gamma+1/2)}, \tag{3.4}$$

where

$$B_N=\frac{2^{2-N}}{\sqrt{N}}\,\frac{\tan\omega\gamma\cos\pi\gamma}{\sin\omega(1-2\gamma)} \prod_{k=1}^{N-1} \frac{1}{\sin\omega(\gamma+(k-1)/2)}, \tag{3.5}$$

and $\omega=\pi/N$. Note that we used an elementary trigonometric identity

$$\prod_{k=1}^{N-1} \sin\frac{\omega k}{2}=\prod_{k=1}^{N-1} \cos\frac{\omega k}{2}=2^{1-N}\sqrt{N} \tag{3.6}$$

in order to derive the expression (3.5) from (2.13).

The form (3.4) shows directly that indeed $w_s>0$ provided the restriction (3.1) is fulfilled. The normalization condition

$$\sum_{s=0}^{N-1} w_s=1 \tag{3.7}$$

leads to a (perhaps new) trigonometric identity containing one parameter γ.

Note that in the limit $\gamma\to 0$ (this transition is needed because formally α_n is not defined for $\gamma=0$) we obtain well-known finite-dimensional Chebyshev polynomials on the circle with the recurrence parameters $\alpha_n = 0$, $n = 0, 1, ..., N-2$; $\alpha_{N-1} = 1$. These polynomials have a trivial weight function $w_s = 1/N$ which can be obtained from (3.4) by the above mentioned limiting transition.

Note also that in the limit $N \to \infty$ we get circle analogs of the ultraspherical polynomials (see, e.g., [12, 14]) having the recurrence parameters

$$\alpha_n = -\frac{\gamma}{n+\gamma+1}.$$

In this section we considered some one-parameter class of the circle analogs of the Askey–Wilson polynomials for q a root of unity. Generic (4-parameter) circle analogues of the Askey–Wilson polynomials (for real values of q) are studied in [23].

4. EXCEPTIONAL CASES OF PASI POLYNOMIALS

So far, we considered the case when a^N, b^N, $(ab)^N \neq 1$. These conditions are needed in order for the recurrence parameters α_n, β_n and explicit expressions (2.2) to be defined for all $n = 0, 1, ..., N-1$. What happens when these conditions are violated?

In this section we consider an exceptional case when the condition

$$ab = q^{-K}, \qquad K = 2, 3, ..., N-1 \tag{4.1}$$

is valid. We can choose b as arbitrary complex parameter such that $b^N \neq 1$.

From (4.1) and (2.1) we conclude that $h_K = 0$, hence the maximal order of the polynomials should be $K = N - m$. The corresponding weight function is written as (see (1.11))

$$w_s = \frac{h_{K-1}}{Q_{K-1}(1/z_s)\, P'_K(z_s)}, \tag{4.2}$$

where z_s, $s = 0, 1, ..., K-1$ are the zeros of the polynomial $P_K(z)$. Recall that orthogonality of the corresponding polynomials is valid only if the zeros z_s are simple. Indeed it is easy to find these zeros directly from the explicit expression (2.2) using the q-binomial theorem,

$$z_s = bq^{s+1/2}, \qquad s = 0, 1, ..., K-1. \tag{4.3}$$

For arbitrary b the zeros z_s are simple and lie at the K vertices of regular N-gon with radius $|b|$.

For the $P'_K(z_s)$ one can easily obtain the expression

$$P'_K(z_s) = b^{K-1} q^{-s+(K-1)/2} \frac{(q;q)_{K-1}\,(q;q)_s}{(q^{-L};q)_s}. \tag{4.4}$$

For $Q_{K-1}(1/z_s)$, using the q-analog of the Chu–Vandermonde formula we can derive the expression

$$Q_{K-1}(1/z_s) = (-b)^{-K+1}\, q^{(1-K^2)/2} \frac{(q;q)_{K-1}\,(bq)^s}{(bq;q)_{K-1}}. \tag{4.5}$$

Combining these results, we get the final formula for the weight function

$$w_s = \frac{(q^{-K+1};q)_s\, b^{-s}}{(b^{-1}q^{-K+1};q)_{K-1}\,(q;q)_s}, \qquad s = 0, 1, ..., K-1. \tag{4.6}$$

It is interesting to note that the normalization condition for the weight function (4.6) is nothing else than the q-binomial theorem [11],

$$\sum_{s=0}^{K-1} \frac{(q^{-K+1};q)_s\, b^{-s}}{(q;q)_s} = (b^{-1}q^{-K+1};q)_{K-1}. \tag{4.7}$$

Consider a special case when $a = b = -q^{-K/2}$. Then we have symmetric polynomials $(P_n(z) = Q_n(z))$ with the reflection parameters

$$\alpha_n = -\frac{\cos(\omega K/2)}{\cos\omega(n+1-K/2)}, \tag{4.8}$$

where $\omega = \pi M/N$.

For the weight function in this case we have

$$w_s = A_K \prod_{k=1}^{s} \frac{\sin\omega(K-k)}{\sin\omega k}, \tag{4.9}$$

where

$$A_{2j+1} = 4^{-j} \prod_{m=1}^{j} \cos^{-2}\omega(j-m+1/2); \qquad A_{2j} = 2^{-1}4^{1-j} \prod_{m=1}^{j-1} \cos^{-2}\omega m. \tag{4.10}$$

When $M = 1$ then $|\alpha_n| < 1$, $n = 0, 1, ..., K-1$ and hence the weight function is positive on the unit circle as is seen from (4.9). However, there are other possibilities for choosing M leaving the weight w_s positive. For example, if $M < N/K$ then obviously w_s is still positive. Moreover we can take the limit $N, M \to \infty$ fixing K. Then we get the PASI polynomials having the same reflection parameters (4.8) but with *arbitrary real* $\omega < \pi/K$.

For $K=N-1$, $M=1$ we have a very simple formula for the weight function

$$w_s = \tan\frac{\omega}{2}\sin\omega s, \qquad s=1, 2, ..., N-1. \tag{4.11}$$

Corresponding polynomials are orthogonal on the vertices of the regular N-gon $z_s = q^s$, $s=1, 2, ..., N-1$ with the only excluded point $z=1$. Note that despite the simple expression for the weight function (4.11) the corresponding polynomials $P_n(z)$ have a non-elementary expression (2.2).

For $K=N-2$, $M=1$ we obtain a weight function of the form

$$w_s = \frac{2}{N\cos\omega}\sin\omega s\sin\omega(s+1), \qquad s=1, 2, ..., N-2. \tag{4.12}$$

In this case the polynomials $P_n(z)$ have the elementary expression

$$P_n(z) = \frac{q^{n/2}}{1-q^{n+1}}\left(\frac{(zq^{-1/2})^{n+1}-1}{zq^{-1/2}-1} - q\,\frac{(zq^{1/2})^{n+1}-1}{zq^{1/2}-1}\right). \tag{4.13}$$

5. APPENDIX

In this Appendix we prove Lemma 1. From the definition of the sum (2.9) we easily derive the recurrence relation

$$S(s, j) = \frac{S(s, j-1) - aq^j S(s-1, j-1)}{1-aq^j}. \tag{5.1}$$

Hence, evaluation of $S(s, j)$ can be reduced to evaluation of $S(s, -1)$.

For the $S(s, -1)$ we have

$$S(s, -1) = {}_2\Phi_1\left(\begin{matrix} q, a \\ q^2/a \end{matrix}; q; q^{1/2-s}/a\right). \tag{5.2}$$

Now observe from the theory of q-ultraspherical polynomials (see, e.g., [3, 11]) that

$$\begin{aligned} &{}_2\Phi_1\left(\begin{matrix} q^{-n}, a \\ q^{1-n}/a \end{matrix}; q; q^{1/2-s}/a\right) \\ &\quad = p^{-n(s+1)}\frac{(a^2; q)_n\,(-p; p)_n}{(a; q)_n\,(-a; p)_n}\,{}_4\Phi_3\left(\begin{matrix} p^{-n}, p^{-s}, p^{s+1}, ap^n \\ -(ap)^{1/2}, (ap)^{1/2}, -p \end{matrix}; p; p\right), \end{aligned} \tag{5.3}$$

where $p = q^{1/2} = \exp(i\pi M/N)$, M is odd.

Substituting $n=N-1$ into (5.3) we see that (as $p^{-N+1}=-p$) the right-hand side of (5.3) is reduced to balanced ${}_3\Phi_2$ series which can be evaluated using the q-Saalschützian formula [11]. So we get the expression

$$S(s,\,-1)=-p\,\frac{(-p;\,p)_{N-1}\,(a;\,p)_{N-1}}{(a;\,q)_{N-1}}\,\frac{a^s(p^3/a;\,q)_s}{(ap;\,q)_s}. \tag{5.4}$$

Then the final formula (2.10) can be proven by induction using (5.1). Note that this expression can be further simplified using an obvious identity,

$$(a;\,q)_{N-1}=\frac{1-a^N}{1-a/q}. \tag{5.5}$$

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